Tracking and regulation in the behavioral framework

Fiaz, Shaik; Takaba, K.; Trentelman, H.L.

Published in:
Automatica

DOI:
10.1016/j.automatica.2011.08.025

Automatica 47 (2011) 2343–2354

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Tracking and regulation in the behavioral framework[☆]

Shaik Fiaz[a,1], K. Takaba[b], H.L. Trentelman[a,1]

[a] Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O Box 407, 9700 AK Groningen, The Netherlands

[b] Department of Applied Mathematics and Physics, Kyoto University, Kyoto 606-8501, Japan

ARTICLE INFO

Article history:
Received 22 June 2010
Received in revised form
24 December 2010
Accepted 28 April 2011
Available online 1 September 2011

Keywords:
Tracking and regulation
Control by interconnection
Internal model
Stabilization

ABSTRACT

Given a plant together with an exosystem generating the disturbances and the reference signals, the problem of asymptotic tracking and regulation is to find a controller such that the plant variable tracks the reference signal regardless of the disturbance acting on the system. If a controller achieves this design objective, we call it a regulator for the plant with respect to the given exosystem. In this paper, we formulate the asymptotic tracking and regulation problem in the behavioral framework, with control by interconnection.

1. Introduction

This paper deals with control in a behavioral context. We consider the problem of finding an admissible, stabilizing controller that regulates the tracking error to zero in the presence of a class of exogenous inputs. In other words, we consider the problem of *asymptotic tracking and regulation* in the behavioral framework.

In the behavioral framework, controlling a plant means restricting its behavior to a desired subset. This restriction is brought about by interconnecting the plant with a controller that we design. The restricted behavior is then called the controlled behavior, which is required to satisfy the design specifications. In terms of representations, control means that additional laws (e.g., in the form of differential equations representing the controller) are imposed on some of the plant variables. Thus, the plant and controller are interconnected through some of their variables. In our context, we do not distinguish between inputs and outputs and we do not restrict ourselves to feedback control. This idea was introduced by Willems (1997) in the context of stabilization and pole placement. In this paper, we use these ideas to solve the problem of asymptotic tracking and regulation.

The problem of asymptotic tracking and regulation has been studied before in the literature, in an input–output framework. See for instance Davison (1975), Davison and Goldenberg (1975), Francis (1977) and Francis and Wonham (1975). The theory has also been extended to nonlinear systems by Isidori and Byrnes (1990). Many results have been collected by Saberi et al. in the book Saberi, Stoorvogel, and Sannuti (2000) (see also Trentelman, Stoorvogel, and Hautus (2001). In these, the concept of internal model principle plays a pivotal role in obtaining a solution to the asymptotic tracking and regulation problem. According to the internal model principle, in order to achieve regulation the controlled system must contain the dynamics of the exosystem.

Our work can be seen as the behavioral generalization of Davison and Goldenberg (1975), Francis (1977) and Francis and Wonham (1975). We use polynomial kernel representations of the plant (see Polderman & Willems, 1997) without input–output considerations. This problem was initially studied by Takaba (2009). In the work of Takaba, only *necessary* conditions were obtained for the existence of a regulator. In Fiaz, Takaba, and Trentelman (2010) necessary and sufficient conditions were obtained. It was assumed that the underlying exosystem is anti-stable and that the underlying plant does not annihilate any signal generated by the exosystem. In this paper, we generalize these results to the case when the underlying exosystem can be any autonomous system (not necessarily anti-stable) and the underlying plant might annihilate signals generated by the exosystem. Necessary and sufficient conditions for the existence of suitable controllers are expressed in terms of the plant and the exosystem. Also, a procedure to construct such controllers is given using the polynomial matrices appearing in the kernel representations of the plant and the exosystem.

[☆] The material in this paper was partially presented at the 49th IEEE Conference on Decision and Control, December 15–17, 2010, Atlanta, Georgia, USA, and the 19th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2010), July 5–9, 2010, Budapest, Hungary. This paper was recommended for publication in revised form by Associate Editor Maria Elena Valcher under the direction of Editor Roberto Tempo.

E-mail addresses: s.fiaz@math.rug.nl (S. Fiaz), takaba@amp.i.kyoto-u.ac.jp (K. Takaba), h.l.trentelman@math.rug.nl (H.L. Trentelman).

[1] Tel.: +31 50 3633999; fax: +31 50 3633800.

0005-1098/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2011.08.025

A few words about the notation and nomenclature used. We use standard symbols for the fields of real and complex numbers $\mathbb{R}$ and $\mathbb{C}$. $\mathbb{C}^-$, and $\bar{\mathbb{C}}_+$ will denote the open left half plane and closed right half plane, respectively. We use $\mathbb{R}^{\mathtt{n}}$, $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, etc., for the real linear spaces of vectors and matrices with components in $\mathbb{R}$.

$\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ denotes the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$. $\mathbb{R}[\xi]$ denotes the ring of polynomials in the indeterminate ξ with real coefficients. We use $\mathbb{R}[\xi]^{\mathtt{n}}$, $\mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{m}}$, for the spaces of vectors and matrices with components in $\mathbb{R}[\xi]$. Elements of $\mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{m}}$ are called *real polynomial matrices*.

We use the notation $\det(A)$ to denote the determinant of a square matrix A. A square, nonsingular real polynomial matrix R is called unimodular if $\det(R)$ is a non-zero constant. It is called *Hurwitz* if all roots of $\det(R)$ lie in the open left half complex plane $\mathbb{C}^-$. It is called *anti-Hurwitz* if all roots of $\det(R)$ lie in the closed right half complex plane $\bar{\mathbb{C}}^+$.

2. Linear differential systems and polynomial kernel representations

In the behavioral approach to linear systems, a continuous time dynamical system is defined by a triple $\sum = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$, where $\mathbb{R}$ is the time axis, $\mathbb{R}^{\mathtt{w}}$ is the signal space, and the *behavior* $\mathfrak{B}$ is a linear subspace of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. Such a triple is called a *linear differential system*. More precisely, there exist a positive integer $\mathtt{g}$ and a polynomial matrix $R \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{w}}$ such that

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \right\}.$$

The set of linear differential systems with manifest variable w taking its value in $\mathbb{R}^{\mathtt{w}}$ is denoted by $\mathfrak{L}^{\mathtt{w}}$.

Let $R \in \mathbb{R}[\xi]^{\mathtt{g}\times\mathtt{w}}$ be a polynomial matrix. If the behavior $\mathfrak{B}$ is represented by $\mathbb{R}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ then we call this a *kernel representation* of $\mathfrak{B}$. Further, a kernel representation is said to be *minimal* if every other kernel representation of $\mathfrak{B}$ has at least grows. A given kernel representation, $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$, is minimal if and only if the polynomial matrix R has full row rank (see Polderman & Willems, 1997, Theorem 3.6.4). The number of rows in any minimal polynomial kernel representation of $\mathfrak{B}$ is equal to the *output cardinality* of $\mathfrak{B}$, denoted by $p(\mathfrak{B})$. This number corresponds to the number of outputs in any input/output representation of $\mathfrak{B}$. We speak of a system as the behavior $\mathfrak{B}$, one of whose representations is given by $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ or just $\mathfrak{B} = \ker(R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right))$. It was shown in Polderman and Willems (1997) that $R_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ and $R_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$ both represent $\mathfrak{B}$ minimally if and only if there exists a unimodular matrix U such that $R_1 = UR_2$.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable w partitioned as $w = (w_1, w_2)$ and let $R_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w_1 + R_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w_2 = 0$ be a minimal representation of $\mathfrak{B}$. We say that w_2 is *free in* $\mathfrak{B}$ if, for any $w_2 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2})$, there exists $w_1 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1})$ such that $(w_1, w_2) \in \mathfrak{B}$. We have w_2 free in $\mathfrak{B}$ if and only if R_1 has full row rank (see Polderman & Willems, 1997). A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called autonomous if there are no free variables, equivalently its output cardinality is equal to $\mathtt{w}$. We denote the set of all autonomous linear differential systems with $\mathtt{w}$ variables by $\mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$. An autonomous behavior $\mathfrak{B}$ is said to be *stable*, if we have $\lim_{t\to\infty} w(t) = 0$ for all $w \in \mathfrak{B}$ and *anti-stable* if for all non-zero $w \in \mathfrak{B}$ we have either $\lim_{t\to\infty} w(t) \neq 0$ or $\lim_{t\to\infty} w(t)$ does not exist. If $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is represented minimally by $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$, then $\mathfrak{B}$ is autonomous if and only if R is square and nonsingular, $\mathfrak{B}$ is stable if and only if R is Hurwitz and $\mathfrak{B}$ is anti-stable if and only if R is anti-Hurwitz.

Definition 2.1. A function of the form

$$h(t) = \sum_{i=1}^{N} p_i(t)e^{a_i t}\cos(b_i t) + q_i(t)e^{a_i t}\sin(b_i t),$$

with p_i, q_i real vector valued polynomials in the indeterminate t, and $a_i, b_i \in \mathbb{R}$, is called a *Bohl function*. A Bohl function $h(t)$ is called *stable Bohl* if in addition $\lim_{t\to\infty} h(t) = 0$. A nonzero Bohl function $h(t)$ is called *anti-stable Bohl* if we have either $\lim_{t\to\infty} h(t) \neq 0$ or $\lim_{t\to\infty} h(t)$ does not exist.

It follows immediately from Polderman and Willems (1997, Theorem 3.2.16) that $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is autonomous if and only if every $w \in \mathfrak{B}$ is a Bohl function, and that $\mathfrak{B}$ is stable if and only if every $w \in \mathfrak{B}$ is a stable Bohl function. Also, $\mathfrak{B}$ is anti-stable if and only if every nonzero $w \in \mathfrak{B}$ is an anti-stable Bohl function.

The next proposition which states that every autonomous behavior can be written as a direct sum of a stable and an anti-stable behavior follows immediately from results in Bisiacco and Valcher (2001a,b) (also see Proposition 2.6.8 in Fiaz, 2010).

Proposition 2.2. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$. Then there exists a stable $\mathfrak{B}_s \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$, and an anti-stable $\mathfrak{B}_a \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$ such that $\mathfrak{B} = \mathfrak{B}_s \oplus \mathfrak{B}_a$.*

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable w partitioned as $w = (w_1, w_2)$. Assume that the first component w_1 is viewed as an observed variable, and the second component w_2 as a to-be-deduced variable. In such systems we can talk about observability. We say that w_2 is *observable* from w_1 in $\mathfrak{B}$ if, whenever (w_1, w_2), $(w_1, w_2') \in \mathfrak{B}$, then $w_2 = w_2'$. The weaker notion of detectability is defined along similar lines. We say that w_2 is *detectable* from w_1 in $\mathfrak{B}$ if, whenever $(w_1, w_2), (w_1, w_2') \in \mathfrak{B}$, then $\lim_{t\to\infty}(w_2 - w_2')(t) = 0$. If $R_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w_1 + R_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w_2 = 0$ is a minimal representation of $\mathfrak{B}$, then w_2 is observable from w_1 in $\mathfrak{B}$ if and only if $R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$ and w_2 is detectable from w_1 in $\mathfrak{B}$ if and only if $R_2(\lambda)$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$ (see Polderman & Willems, 1997).

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1, w_2). Often we are interested only in the behavior of one of the components, say the variable w_1, obtained by projecting $\mathfrak{B}$ onto this component. This behavior $(\mathfrak{B})_{w_1}$ is defined by $(\mathfrak{B})_{w_1} := \{w_1 \mid \exists w_2 \text{ such that } (w_1, w_2) \in \mathfrak{B}\}$. Starting with a polynomial kernel representation of $\mathfrak{B}$, in the following proposition we give a procedure for obtaining a polynomial kernel representation for $(\mathfrak{B})_{w_1}$ (see Polderman & Willems, 1997).

Proposition 2.3. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1, w_2) be represented by $R_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w_1 + R_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w_2 = 0$. Let U be a unimodular matrix such that $UR_2 = \begin{bmatrix} R_{12} \\ 0 \end{bmatrix}$ with R_{12} full row rank. Let UR_1 be partitioned correspondingly into $UR_1 = \begin{bmatrix} R_{11} \\ R_{21} \end{bmatrix}$. Then akernel representation of $(\mathfrak{B})_{w_1}$ is given by $R_{21}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0$.*

3. Review of stabilization by interconnection

In this section we will briefly recall the notion of stabilization by interconnection. We will first look at the full interconnection case, i.e. the case when all the plant variables are available for interconnection.

Definition 3.1. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ be a plant behavior. A controller for $\mathcal{P}$ is a system behavior $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$. The full interconnection of $\mathcal{P}$ and $\mathcal{C}$, shown schematically in Fig. 1, is defined as the system with behavior $\mathcal{P} \cap \mathcal{C}$. This behavior is called the *controlled behavior*, and is also an element of $\mathfrak{L}^{\mathtt{w}}$. The full interconnection is called *regular* if $p(\mathcal{P} \cap \mathcal{C}) = p(\mathcal{P}) + p(\mathcal{C})$. In that case we call $\mathcal{C}$ a *regular controller*.

In full interconnection, the regularity condition is equivalent to: $\mathcal{C}$ does not re-impose restrictions on the plant variable w that are already present in the laws of $\mathcal{P}$ (see Willems, 1997).

A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *stabilizable* if for every $w \in \mathfrak{B}$ there exists $w' \in \mathfrak{B}$ such that $w'(t) = w(t)$ for $t \leq 0$,

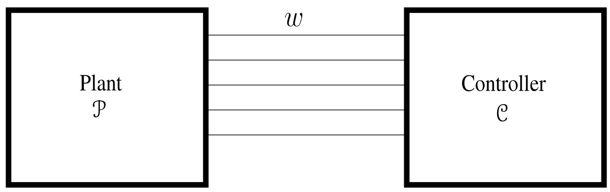


Fig. 1. Full interconnection of plant and controller.

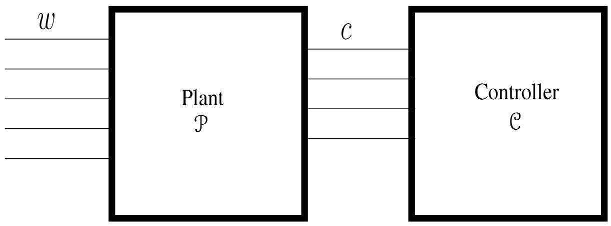


Fig. 2. Partial interconnection of plant and controller.

and $\lim_{t\to\infty} w'(t) = 0$. If $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is represented minimally by $R\left(\frac{d}{dt}\right) w = 0$, then $\mathfrak{B}$ is stabilizable if and only if $R(\lambda)$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^{+}$. A given plant is stabilizable if and only if we can stabilize it by interconnecting it with a suitable controller, called a stabilizing controller, which is defined as follows (Willems & Trentelman, 2002).

Definition 3.2. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$. A controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ is said to be a *stabilizing controller* if the behavior $\mathcal{P} \cap \mathcal{C}$ is stable and the interconnection is regular.

The following result is shown in Willems (1997).

Proposition 3.3. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$. Then the following statements are equivalent:*

(1) *$\mathcal{P}$ is stabilizable,*
(2) *there exists a stabilizing controller for $\mathcal{P}$.*

Next we will look at the so called partial interconnection case, in which only a pre-specified subset of the plant variables is available for interconnection. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be a linear differential system, with system variable (w, c), where w takes its values in $\mathbb{R}^{\mathtt{w}}$ and c in $\mathbb{R}^{\mathtt{c}}$. The variable w should be interpreted as the variable to be controlled, the variable c as the one through which we can interconnect the plant with a controller, called the control variable. Let $\mathbb{C} \in \mathfrak{L}^{\mathtt{c}}$ (to be interpreted as a controller behavior) with variable c.

Definition 3.4. The interconnection of $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ through c, shown schematically in Fig. 2, is defined as the system behavior $\mathcal{P} \wedge_c \mathcal{C} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, given by $\mathcal{P} \wedge_c \mathcal{C} = \{(w, c) \mid (w, c) \in \mathcal{P} \text{ and } c \in \mathbb{C}\}$. The behavior $\mathcal{P} \wedge_c \mathcal{C}$ is called *the full controlled behavior*. The behavior $(\mathcal{P} \wedge_c \mathcal{C})_w \in \mathfrak{L}^{\mathtt{w}}$ that is obtained by eliminating c from $\mathcal{P} \wedge_c \mathcal{C}$ is called *the manifest controlled behavior*. The interconnection of $\mathcal{P}$ and $\mathcal{C}$ through c is called *regular* if $\mathrm{p}(\mathcal{P} \wedge_c \mathcal{C}) = \mathrm{p}(\mathcal{P}) + \mathrm{p}(\mathcal{C})$. $\mathcal{C}$ is then called a *regular controller*.

In the partial interconnection case, the regularity condition is equivalent to: $\mathcal{C}$ does not re-impose restrictions on the control variable c that are already present in the laws of $\mathcal{P}$ (see Belur & Trentelman, 2002 and Belur, 2003).

Given $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable (w_1, w_2), in this paper we use the notation $\mathcal{N}_{w_1}(\mathcal{P})$ to indicate the behavior obtained by putting $w_2 = 0$ and projecting onto the variable w_1 i.e., $\mathcal{N}_{w_1}(\mathcal{P}) = \{w_1 \mid (w_1, 0) \in \mathcal{P}\}$.

In the next section we will formulate the asymptotic tracking and regulation problem studied in this paper.

4. Asymptotic tracking and regulation

For a given plant behavior with its to-be-controlled variable w and reference signal r, an important synthesis problem in control is to design a controller such that the plant variable w follows the reference signal r in the resulting system after interconnecting the plant and the controller. This is called the *asymptotic tracking problem*. A classical approach to this problem is to let the reference signal be generated by an autonomous system called the *exosystem*. One then incorporates the dynamics of the exosystem into the dynamics of the plant and defines a new variable e as the difference between the reference signal r and w. The asymptotic tracking problem is then reformulated as: design a controller that, after interconnection with the plant, drives the signal e to zero.

A second important synthesis problem is the problem of *regulation*. For a given plant with to-be-controlled variable w, and external disturbance acting on the plant (which is assumed to be free in the plant), the problem here is to design a controller such that in the resulting system after interconnection of the plant and the controller, the disturbance remains free and the plant variable w converges to zero as time tends to infinity, regardless of the disturbance acting on the plant. A controller such that after interconnection with the plant, the disturbance remains free is called an *admissible controller*. In line with the approach to the regulation problem in Francis (1977) and Francis and Wonham (1975) and similarly to the asymptotic tracking problem given above, we approach this problem by assuming the disturbance to be generated by some linear time invariant autonomous system, again called the *exosystem*. Then one incorporates the dynamics of the exosystem into the dynamics of the plant, and requires the variable w in this interconnected system to converge to zero as time tends to infinity.

Combining these two synthesis problems we can formulate a single new synthesis problem by requiring the design of a controller such that the interconnected system variable tracks a given reference signal, regardless of the disturbance. This is done by combining the two exosystems into a single one and requires regulation of the tracking error.

In addition to the requirements of asymptotic tracking and regulation, a realistic design requires the system to go to rest in the absence of disturbances (i.e., if the disturbance signal is identically equal to zero). An *admissible controller* that takes the system to rest in the absence of disturbances is called a *stabilizing* controller. An admissible controller which achieves all three requirements, i.e. asymptotic tracking, regulation and stabilization, is called a *regulator*.

4.1. Problem formulation

In this subsection we will introduce the problem of asymptotic tracking and regulation in a behavioral context, with control by regular, partial interconnection. We start with a plant behavior $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$, with plant variables (w, c, v), shown schematically in Fig. 3(b). The system variable has been partitioned into w, c and v. These variables represent the to-be-controlled variable (including tracking error), the interconnection variable (such as sensor measurements and actuator inputs), and the external disturbances and reference signals, respectively. The interconnection variable c is the system variable through which we are allowed to interconnect $\mathcal{P}$ with the controller $\mathbb{C} \in \mathfrak{L}^{\mathtt{c}}$. As the components of the variable v represent reference signals and external disturbances, we assume v to be free in $\mathcal{P}$. In addition to the plant $\mathcal{P}$, let an exosystem $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}$ which generates the disturbance and the reference signal be given, as shown schematically in Fig. 3(a).

Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$, shown schematically in Fig. 3(c). Then the interconnection of $\mathcal{P}$ with $\mathbb{C}$ (shown schematically in Fig. 4) is given by

$$\mathcal{P} \wedge_c \mathcal{C} = \{(w, c, v) \mid (w, c, v) \in \mathcal{P} \text{ and } c \in \mathcal{C}\}. \quad (1)$$

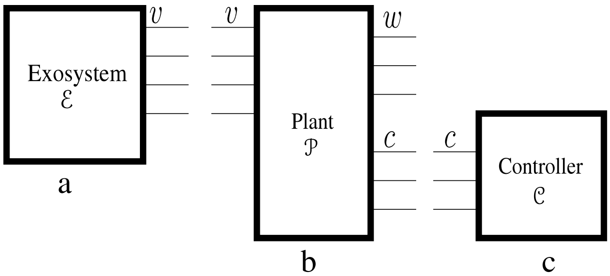


Fig. 3. Exosystem, plant and controller.

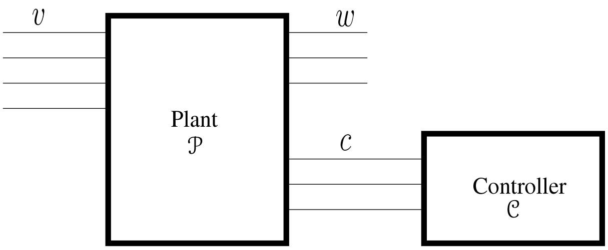


Fig. 4. Interconnection of the plant and the controller.

As v is interpreted as unknown disturbance, it should remain free after interconnecting the plant with a controller. In order to highlight this, we give the following definition:

Definition 4.1. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$. Assume v is free in $\mathcal{P}$. Then $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called an *admissible* controller for $\mathcal{P}$ if v is free in $\mathcal{P} \wedge_c \mathcal{C}$.

In the context of asymptotic tracking and regulation a controller is called stabilizing if, whenever the disturbance v is zero, the to-be-regulated variable w and interconnection variable c tend to zero as time runs off to infinity:

Definition 4.2. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$, with v free. An admissible controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called *stabilizing* if $\lim_{t\to\infty}(w(t), c(t)) = (0, 0)$ for all $(w, c, 0) \in \mathcal{P} \wedge_c \mathcal{C}$ (equivalently, $\mathcal{N}_{(w,c)}(\mathcal{P}) \wedge_c \mathcal{C}$ is stable).

In the following theorem we establish necessary and sufficient conditions on the plant for the existence of a regular, admissible, stabilizing controller:

Theorem 4.3. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$. Assume v is free in $\mathcal{P}$. Then there exists a regular, admissible, stabilizing controller for $\mathcal{P}$ if and only if*

(1) *$\mathcal{N}_{(w,c)}(\mathcal{P})$ is stabilizable, and*
(2) *w is detectable from (c, v) in $\mathcal{P}$.*

Proof. Let $R_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w + R_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) c + R_3\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) v = 0$ be a minimal representation of $\mathcal{P}$. Then there exists a unimodular matrix U such that

$$U\begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \end{bmatrix},$$

where R_{11} has full row rank. Then we have

$$\mathcal{P} = \ker\left(\begin{bmatrix} R_{11}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & R_{12}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & R_{13}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 0 & R_{22}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & R_{23}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix}\right), \tag{2}$$

$$\mathcal{N}_{(w,c)}(\mathcal{P}) = \ker\left(\begin{bmatrix} R_{11}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & R_{12}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 0 & R_{22}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix}\right). \tag{3}$$

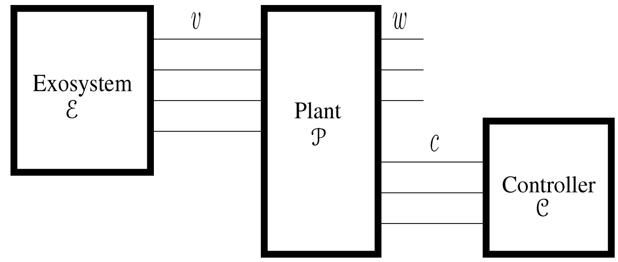


Fig. 5. Interconnection of the plant, controller and the exosystem.

(only if) Let $C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) c = 0$ be a minimal representation of a regular, admissible, stabilizing controller $\mathcal{C}$ for $\mathcal{P}$. Then $\mathcal{P} \wedge_c \mathcal{C}$ is given by

$$\mathcal{P} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_{11}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & R_{12}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & R_{13}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 0 & R_{22}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & R_{23}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 0 & C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & 0 \end{bmatrix}\right). \tag{4}$$

We have

$$\mathcal{N}_{(w,c)}(\mathcal{P}) \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_{11}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & R_{12}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 0 & R_{22}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 0 & C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix}\right). \tag{5}$$

Since v is free in $\mathcal{P} \wedge_c \mathcal{C}$ and $\mathcal{N}_{(w,c)}(\mathcal{P}) \wedge_c \mathcal{C}$ is stable, $\begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & C \end{bmatrix}$ is square, nonsingular and Hurwitz, which in turn implies that R_{11} is square, nonsingular and Hurwitz and $\begin{bmatrix} R_{11}(\lambda) & R_{12}\lambda \\ 0 & R_{22}\lambda \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$. From Eq. (2) w is detectable from (c, v) in $\mathcal{P}$. From Eq. (3) we conclude that $\mathcal{N}_{(w,c)}(\mathcal{P})$ is stabilizable.

(if) From Eq. (2), w is detectable from (c, v) in $\mathcal{P}$ implies that R_{11} is Hurwitz. From Eq. (3), $\mathcal{N}_{(w,c)}(\mathcal{P})$ stabilizable implies that $\begin{bmatrix} R_{11}(\lambda) & R_{12}(\lambda) \\ 0 & R_{22}(\lambda) \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$, which in turn implies that $R_{22}(\lambda)$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$. Choose C such that $\begin{bmatrix} R_{22} \\ C \end{bmatrix}$ is Hurwitz. Then $\begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & C \end{bmatrix}$ is Hurwitz. Define $\mathcal{C} = \ker(C\left(\frac{\mathrm{d}}{\mathrm{d}t}\right))$. Then it is easy to verify that this $\mathcal{C}$ is a regular, admissible, stabilizing controller for $\mathcal{P}$. $\square$

The interconnection of the plant $\mathcal{P}$ with the exosystem $\mathcal{E}$ and controller $\mathcal{C}$ is shown schematically in Fig. 5 and is given by

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \{(w, c, v) \mid (w, c, v) \in \mathcal{P}, v \in \mathcal{E} \text{ and } c \in \mathcal{C}\}. \tag{6}$$

We have the following definition of a regulator.

Definition 4.4. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$. Assume v is free in $\mathcal{P}$. Then $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called a *regulator* for $\mathcal{P}$ with respect to $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}$, if

(1) $\mathcal{C}$ is a regular, admissible, stabilizing controller for $\mathcal{P}$, and
(2) for all $(w, c, v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$ we have $\lim_{t\to\infty} w(t) = 0$, i.e., $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w$ is stable.

Condition (2) in the above definition asks the controller to achieve regulation of the system variable w.

We now formulate the main problem of this paper:

Problem 1. Given a plant $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ with system variable (w, c, v), with v free in $\mathcal{P}$, and an autonomous system $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{aut}}$ with system variable v, find a necessary and sufficient condition for the existence of a regulator $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ for $\mathcal{P}$ with respect to $\mathcal{E}$.

5. Solution to the asymptotic tracking and regulation problem

As a first step in resolving Problem 1, we will show that without loss of generality we can assume that in $\mathcal{P} \wedge_v \mathcal{E}$, the interconnection of plant and exosystem, v is observable from (w, c), equivalently, $\mathcal{E} \cap \mathcal{N}_v(\mathcal{P}) = 0$.

Let

$$R_1\left(\frac{d}{dt}\right) w + R_2\left(\frac{d}{dt}\right) c + R_3\left(\frac{d}{dt}\right) v = 0, \quad \text{and} \tag{7}$$

$$V\left(\frac{d}{dt}\right) v = 0 \tag{8}$$

be minimal representations of $\mathcal{P}$ and $\mathcal{E}$ respectively, where V is square and nonsingular. Factorize

$$\begin{bmatrix} R_3 \\ V \end{bmatrix} = \begin{bmatrix} R'_3 \\ V' \end{bmatrix} D,$$

where D is square and nonsingular and $\begin{bmatrix} R'_3\lambda \\ V'\lambda \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$. Define

$$\mathcal{P}' := \left\{(w, c, v) \mid R_1\left(\frac{d}{dt}\right) w + R_2\left(\frac{d}{dt}\right) c + R'_3\left(\frac{d}{dt}\right) v = 0\right\}, \tag{9}$$

and

$$\mathcal{E}' := \left\{v \mid V'\left(\frac{d}{dt}\right) v = 0\right\}. \tag{10}$$

We have

$$\mathcal{P}' \wedge_v \mathcal{E}' = \ker\left(\begin{bmatrix} R_1\left(\frac{d}{dt}\right) & R_2\left(\frac{d}{dt}\right) & R'_3\left(\frac{d}{dt}\right) \\ 0 & 0 & V'\left(\frac{d}{dt}\right) \end{bmatrix}\right). \tag{11}$$

It is easy to see that v is observable from (w, c) in $\mathcal{P}' \wedge_v \mathcal{E}'$ (use the fact that $\begin{bmatrix} R'_3\lambda \\ V'\lambda \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$).

Let $\mathcal{C} \in \mathfrak{L}^c$. The following theorem shows that for the solvability of Problem 1 the assumption $\mathcal{E} \cap \mathcal{N}_v(\mathcal{P}) = 0$ can indeed be made without loss of generality:

Theorem 5.1. *Let $\mathcal{P}, \mathcal{E}, \mathcal{P}'$ and $\mathcal{E}'$ be given by Eqs. (7)–(10), respectively. Then $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if $\mathcal{C}$ is a regulator for $\mathcal{P}'$ with respect to $\mathcal{E}'$.*

Proof. Let $C\left(\frac{d}{dt}\right) c = 0$ be a minimal representation of $\mathcal{C}$. We have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_1\left(\frac{d}{dt}\right) & R_2\left(\frac{d}{dt}\right) & R_3\left(\frac{d}{dt}\right) \\ 0 & C\left(\frac{d}{dt}\right) & 0 \end{bmatrix}\right), \quad \text{and} \tag{12}$$

$$\mathcal{P}' \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_1\left(\frac{d}{dt}\right) & R_2\left(\frac{d}{dt}\right) & R'_3\left(\frac{d}{dt}\right) \\ 0 & C\left(\frac{d}{dt}\right) & 0 \end{bmatrix}\right). \tag{13}$$

From the above it is easy to see that the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular, v is free in $\mathcal{P} \wedge_c \mathcal{C}$, and $\mathcal{N}_{(w,c)}(\mathcal{P}) \wedge_c \mathcal{C}$ is stable if and only if $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ is square, nonsingular and Hurwitz. In turn, this holds if and only if the interconnection $\mathcal{P}' \wedge_c \mathcal{C}$ is regular, v is free in $\mathcal{P}' \wedge_c \mathcal{C}$, and $\mathcal{N}_{(w,c)}(\mathcal{P}') \wedge_c \mathcal{C}$ is stable. In order to proceed we now show $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w = (\mathcal{P}' \wedge_v \mathcal{E}' \wedge_c \mathcal{C})_w$.

We have

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_1\left(\frac{d}{dt}\right) & R_2\left(\frac{d}{dt}\right) & R'_3D\left(\frac{d}{dt}\right) \\ 0 & C\left(\frac{d}{dt}\right) & 0 \\ 0 & 0 & V'D\left(\frac{d}{dt}\right) \end{bmatrix}\right),$$

and

$$\mathcal{P}' \wedge_v \mathcal{E}' \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_1\left(\frac{d}{dt}\right) & R_2\left(\frac{d}{dt}\right) & R'_3\left(\frac{d}{dt}\right) \\ 0 & C\left(\frac{d}{dt}\right) & 0 \\ 0 & 0 & V'\left(\frac{d}{dt}\right) \end{bmatrix}\right).$$

There exists a unimodular matrix $\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \end{bmatrix}$ such that

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \end{bmatrix} \begin{bmatrix} R_1 & R_2 & R'_3D \\ 0 & C & 0 \\ 0 & 0 & V'D \end{bmatrix} = \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & G_{22} & G_{23}D \end{bmatrix}, \tag{14}$$

and

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \end{bmatrix} \begin{bmatrix} R_1 & R_2 & R'_3 \\ 0 & C & 0 \\ 0 & 0 & V' \end{bmatrix} = \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & G_{22} & G_{23} \end{bmatrix} \tag{15}$$

where $\begin{bmatrix} G_{22} & G_{23} \end{bmatrix}$ and $\begin{bmatrix} G_{22} & G_{23}D \end{bmatrix}$ have full row rank. Hence, from Eqs. (14) and (15) we have

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} G_{11}\left(\frac{d}{dt}\right) & 0 & 0 \\ G_{21}\left(\frac{d}{dt}\right) & G_{22}\left(\frac{d}{dt}\right) & G_{23}D\left(\frac{d}{dt}\right) \end{bmatrix}\right),$$

$$\mathcal{P}' \wedge_v \mathcal{E}' \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} G_{11}\left(\frac{d}{dt}\right) & 0 & 0 \\ G_{21}\left(\frac{d}{dt}\right) & G_{22}\left(\frac{d}{dt}\right) & G_{23}\left(\frac{d}{dt}\right) \end{bmatrix}\right),$$

and $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w = \ker(G_{11}\left(\frac{d}{dt}\right)) = (\mathcal{P}' \wedge_c \mathcal{C} \wedge_v \mathcal{E}')_w$. From the above and using Definitions 4.2 and 4.4 we conclude that $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if $\mathcal{C}$ is a regulator for $\mathcal{P}'$ with respect to $\mathcal{E}'$. □

The following theorem will be instrumental in solving Problem 1.

Theorem 5.2. *Let $\mathcal{K} \in \mathfrak{L}^{w+v}$ with system variable (w, v). Assume v is free in $\mathcal{K}$. Let $\mathcal{E} \in \mathfrak{L}^{v}_{\text{aut}}$ be an anti-stable system with system variable v. Then $(\mathcal{K} \wedge_v \mathcal{E})_w$ is stable if and only if the following conditions hold:*

(1) $\lim_{t\to\infty} w(t) = 0$ for all $(w, 0) \in \mathcal{K}$, i.e., $\mathcal{N}_w(\mathcal{K})$ is stable, and
(2) $(0, v) \in \mathcal{K}$ holds for all $v \in \mathcal{E}$, i.e., $\mathcal{E} \subseteq \mathcal{N}_v(\mathcal{K})$.

Proof. *(if)* $(w, v) \in \mathcal{K} \wedge_v \mathcal{E}$ implies $(w, v) \in \mathcal{K}$ and $v \in \mathcal{E}$. As $(0, v) \in \mathcal{K}$ for all $v \in \mathcal{E}$, from linearity, we have $(w, v) - (0, v) \in \mathcal{K}$. Therefore $(w, 0) \in \mathcal{K}$. Since we have $\lim_{t\to\infty} w(t) = 0$ for all $(w, 0) \in \mathcal{K}$, we conclude that $\lim_{t\to\infty} w(t) = 0$ holds for all $(w, v) \in \mathcal{K} \wedge_v \mathcal{E}$.

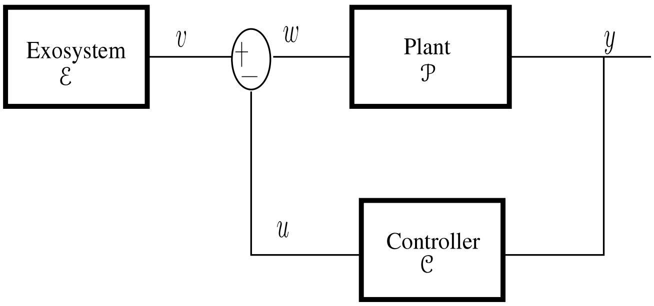


Fig. 6. Interconnection of the plant, controller and the exosystem.

(only if) We have $\{(w,0) \mid (w,0) \in \mathcal{K}\} \subseteq \mathcal{K} \wedge_v \mathcal{E}$. Since $\lim_{t\to\infty} w(t) = 0$ for all $(w,v) \in \mathcal{K} \wedge_v \mathcal{E}$, we obtain $\lim_{t\to\infty} w(t) = 0$ for all $(w,0) \in \mathcal{K}$.

Let $R_1\left(\frac{d}{dt}\right) w + R_2\left(\frac{d}{dt}\right) v = 0$ be a minimal representation of $\mathcal{K}$. Let $v \in \mathcal{E}$. As v is free in $\mathcal{K}$ there exists a w such that

$$R_1\left(\frac{d}{dt}\right) w = -R_2\left(\frac{d}{dt}\right) v. \tag{16}$$

As $(\mathcal{K} \wedge_v \mathcal{E})_w$ is stable, w is a stable Bohl function. Hence, the LHS of Eq. (16) is a stable Bohl function. Also, since $\mathcal{E}$ is anti-stable, v is either identically equal to 0 or anti-stable Bohl. This implies that the RHS of Eq. (16) is either identically equal to 0, or an anti-stable Bohl function. Eq. (16) thus implies that $R_1\left(\frac{d}{dt}\right) w = -R_2\left(\frac{d}{dt}\right) v = 0$. Consequently, $(w,0) \in \mathcal{K}$. From linearity we have $(w,v) - (w,0) \in \mathcal{K}$, which implies that $(0,v) \in \mathcal{K}$. Therefore $v \in \mathcal{N}_v(\mathcal{K})$. □

Remark 5.3. By applying Theorem 5.2 to $\mathcal{K} := (\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}$ we find that an admissible, regular, stabilizing controller $\mathcal{C}$ achieves the regulation condition that $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w$ is stable if and only if $\mathcal{E} \subseteq \mathcal{N}_v((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)})$. Thus, Condition (2) of Theorem 5.2 provides a version of the so called internal model principle in the behavioral setting: in order to achieve regulation of the variable w subject to all exogenous signals $v \in \mathcal{E}$, the controlled behavior $(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}$ must contain the dynamics of $\mathcal{E}$, in the sense that $\mathcal{E} \subseteq \mathcal{N}_v((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)})$. In this way, the behavioral approach to asymptotic tracking and regulation brings forward the 'internal model principle' very clearly and directly.

In the following example we apply the internal model principle to interpret the classical so called type-k systems.

Example 5.4. Let $\mathcal{P} \in \mathfrak{L}^4$ and $\mathcal{E} \in \mathfrak{L}^1$ with system variable $(w,(u,y),v)$ and v be given by

$$\mathcal{P} = \left\{(w,(u,y),v) \mid p\left(\frac{d}{dt}\right) y = q\left(\frac{d}{dt}\right) w,\ w = v - u\right\},$$

$$\mathcal{E} = \left\{v \mid \frac{d^n}{dt^n} v = 0\right\},$$

where $p \neq 0$. All variables w, u, y and v are scalar valued. We assume that p has no roots in 0. Let $\mathcal{C} \in \mathfrak{L}^2$ given by $\mathcal{C} = \{(c_1,c_2) \mid c_1\left(\frac{d}{dt}\right) u + c_2\left(\frac{d}{dt}\right) y = 0\}$ be a controller, where c_1 and c_2 are nonzero coprime polynomials. The interconnection $\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$ is shown schematically in Fig. 6. It can be shown that the controller $\mathcal{C}$ is admissible, regular and stabilizing if and only if the polynomial $qc_2 - pc_1$ is Hurwitz. Let g be the greatest common divisor of p and c_2, and let p' and c_2' be such that $p = gp'$, $c_2 = gc_2'$. Straightforward computation shows that $(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}$ is represented by $(q\left(\frac{d}{dt}\right) c_2'\left(\frac{d}{dt}\right) - p'\left(\frac{d}{dt}\right) c_1\left(\frac{d}{dt}\right)) w = -p'\left(\frac{d}{dt}\right) c_1\left(\frac{d}{dt}\right) v$. After noting that $qc_2' - p'c_1$ is Hurwitz, Theorem 5.2 states that $\mathcal{C}$ is a regulator if and only if there exists $f \in \mathbb{R}[\xi]$ such that $p'(\xi)c_1(\xi) = f(\xi)\xi^n$ (internal model principle). Therefore $\mathcal{C}$ is a regulator if and only if c_1 has at least an n-fold root in 0, equivalently, the controller transfer function has at least an n-fold pole in 0.

As regulation is an asymptotic property, intuitively the stable part of the exosystem does not affect regulation. Indeed, in the following theorem, we show that we can reduce the general problem to the case when the exosystem is anti-stable.

Theorem 5.5. *Let $\mathcal{P} \in \mathfrak{L}^{w+c+v}$ and $\mathcal{E} \in \mathfrak{L}^{v}_{aut}$. Assume v is free in $\mathcal{P}$. Let $\mathcal{E} = \mathcal{E}_s \oplus \mathcal{E}_a$ where $\mathcal{E}_s \in \mathfrak{L}^{v}_{aut}$ is stable and $\mathcal{E}_a \in \mathfrak{L}^{v}_{aut}$ is anti-stable. Let $\mathcal{C} \in \mathfrak{L}^{c}$. Then the following statements are equivalent.*

(1) *$\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.*
(2) *$\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}_a$.*

Proof. Before turning to the actual proof of this theorem, we will first prove the following three lemmas. □

Lemma 5.6. *Let $\mathcal{P} \in \mathfrak{L}^{w+v}$, $\mathcal{E} \in \mathfrak{L}^{v}_{aut}$. Assume v is free in $\mathcal{P}$. Let $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ with $\mathcal{E}_1, \mathcal{E}_2 \in \mathfrak{L}^{v}_{aut}$. Then*

$$(\mathcal{P} \wedge_v \mathcal{E}_1) + (\mathcal{P} \wedge_v \mathcal{E}_2) = (\mathcal{P} \wedge_v \mathcal{E}).$$

Proof. As $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ the inclusion $(\mathcal{P} \wedge_v \mathcal{E}_1) + (\mathcal{P} \wedge_v \mathcal{E}_2) \subseteq (\mathcal{P} \wedge_v \mathcal{E})$ is straightforward. To prove the converse inclusion let $(w,v) \in \mathcal{P} \wedge_v \mathcal{E}$. Then there exist $v_1 \in \mathcal{E}_1$ and $v_2 \in \mathcal{E}_2$ such that $v = v_1 + v_2$. Since v is free in $\mathcal{P}$, there exists w_1 such that $(w_1,v_1) \in \mathcal{P} \wedge_v \mathcal{E}_1 \subseteq \mathcal{P} \wedge_v \mathcal{E}$. Define $w_2 := w - w_1$. By linearity, we have $(w_2,v_2) = (w,v) - (w_1,v_1) \in \mathcal{P} \wedge_v \mathcal{E} \subseteq \mathcal{P}$. Moreover, $(w_2,v_2) \in \mathcal{P} \wedge_v \mathcal{E}_2$ since $v_2 \in \mathcal{E}_2$. Consequently, $(w,v) = (w_1,v_1) + (w_2,v_2) \in (\mathcal{P} \wedge_v \mathcal{E}_1) + (\mathcal{P} \wedge_v \mathcal{E}_2)$. This implies $(\mathcal{P} \wedge_v \mathcal{E}_1) + (\mathcal{P} \wedge_v \mathcal{E}_2) \supseteq \mathcal{P} \wedge_v \mathcal{E}$. □

Lemma 5.7. *Let $\mathcal{P} \in \mathfrak{L}^{w+v}$ and let $\mathcal{E}_s \in \mathfrak{L}^{v}_{aut}$ be stable. If $\mathcal{N}_w(\mathcal{P})$ is stable then $\mathcal{P} \wedge_v \mathcal{E}_s$ is stable.*

Proof. Let $R_1\left(\frac{d}{dt}\right) w + R_2\left(\frac{d}{dt}\right) v = 0$ and $S\left(\frac{d}{dt}\right) v = 0$ be minimal representations of $\mathcal{P}$ and $\mathcal{E}_s$ respectively, where S is Hurwitz. We have

$$\mathcal{P} \wedge_v \mathcal{E}_s = \ker\left(\begin{bmatrix} R_1\left(\frac{d}{dt}\right) & R_2\left(\frac{d}{dt}\right) \\ 0 & S\left(\frac{d}{dt}\right) \end{bmatrix}\right) \tag{17}$$

and

$$\mathcal{N}_w(\mathcal{P}) = \ker\left(R_1\left(\frac{d}{dt}\right)\right). \tag{18}$$

The stability of $\mathcal{N}_w(\mathcal{P})$ implies that $R_1(\lambda)$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$, which in turn implies that $\begin{bmatrix} R_1(\lambda) & R_2(\lambda) \\ 0 & S(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. Therefore $\mathcal{P} \wedge_v \mathcal{E}_s$ stable. □

Lemma 5.8. *Let $\mathcal{P} \in \mathfrak{L}^{w+c+v}$ and $\mathcal{E} \in \mathfrak{L}^{v}_{aut}$. Assume v is free in $\mathcal{P}$. Let $\mathcal{E} = \mathcal{E}_s \oplus \mathcal{E}_a$ where $\mathcal{E}_s \in \mathfrak{L}^{v}_{aut}$ is stable and $\mathcal{E}_a \in \mathfrak{L}^{v}_{aut}$ is anti-stable. Let $\mathcal{C} \in \mathfrak{L}^{c}$ be such that v is free in $\mathcal{P} \wedge_c \mathcal{C}$. Then the following statements are equivalent:*

(1) *$(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w$ is stable.*
(2) *$(\mathcal{P} \wedge_v \mathcal{E}_a \wedge_c \mathcal{C})_w$ is stable.*

Proof. ((1) ⇒ (2)) As $\mathcal{E}_a \subseteq \mathcal{E}$ we have $\mathcal{P} \wedge_v \mathcal{E}_a \wedge_c \mathcal{C} \subseteq \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$ which implies $(\mathcal{P} \wedge_v \mathcal{E}_a \wedge_c \mathcal{C})_w \subseteq (\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w$. Therefore, the stability of $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w$ implies that $(\mathcal{P} \wedge_v \mathcal{E}_a \wedge_c \mathcal{C})_w$ is stable.

((2) ⇒ (1)) We have $(\mathcal{P} \wedge_v \mathcal{E}_a \wedge_c \mathcal{C})_w = ((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E}_a)_w$ stable. From Theorem 5.2 we must have the stability of $\mathcal{N}_w$

$((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)})$. As v is free in $\mathcal{P} \wedge_c \mathcal{C}$, it is easy to see that v is free in $(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}$. Therefore, from Lemma 5.6 we have $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{(w,v)} = (\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E} = (\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E}_s + (\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E}_a$. This implies that $((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E})_w = ((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E}_s)_w + ((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E}_a)_w$. Using that $\mathcal{N}_w((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)})$ is stable and Lemma 5.7 we have that $(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E}_s$ is stable, which implies that $((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E}_s)_w$ is stable. From the above we conclude that $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w = ((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E})_w$ is stable. □

Finally, by combining these lemmas we arrive at:

Proof of Theorem 5.5. It is evident from Lemma 5.8 and Definition 4.4 that $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if $\mathcal{C}$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}_a$. □

Based on Theorems 5.1 and 5.5, without loss of generality we hereafter make the following assumptions:

Assumptions. A1. $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\text{aut}}$ is an anti-stable system, and
A2. v is observable from (w, c) in $\mathcal{P} \wedge_v \mathcal{E}$, i.e., $\mathcal{E} \cap \mathcal{N}_v(\mathcal{P}) = 0$.

The following theorem is the main result of this paper. It provides a complete solution to Problem 1.

Theorem 5.9. *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{v}}$ with system variable (w, c, v). Assume v is free in $\mathcal{P}$. Let $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\text{aut}}$ with system variable v. Assume $\mathcal{E}$ is anti-stable and v is observable from (w, c) in $\mathcal{P} \wedge_v \mathcal{E}$. Then there exists a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if the following conditions hold:*

(1) *(w, v) is detectable from c in $\mathcal{P} \wedge_v \mathcal{E}$,*
(2) *$\mathcal{N}_{(w,c)}(\mathcal{P})$ is stabilizable, and*
(3) *there exists a polynomial matrix $X \in \mathbb{R}[\xi]^{\mathtt{c}\times\mathtt{v}}$ such that $(0, X\left(\frac{d}{dt}\right) v, v) \in \mathcal{P}$ for all $v \in \mathcal{E}$.*

Proof. Let

$$R_1\left(\frac{d}{dt}\right) w + R_2\left(\frac{d}{dt}\right) c + R_3\left(\frac{d}{dt}\right) v = 0, \quad \text{and} \tag{19}$$

$$V\left(\frac{d}{dt}\right) v = 0 \tag{20}$$

be minimal representations of $\mathcal{P}$ and $\mathcal{E}$, respectively.
(only if)

(1) We easily see that $\{(w, 0, v) \mid (w, 0, v) \in \mathcal{P} \wedge_v \mathcal{E}\} \subseteq \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$. It then follows from Definition 4.4 that $\lim_{t\to\infty} (w(t), 0) = 0$ for all $(w, 0, v) \in \mathcal{P} \wedge_v \mathcal{E}$. Hence, if $(w, 0, v) \in \mathcal{P} \wedge_v \mathcal{E}$ then w is a stable Bohl function. As v is observable from (w, c) in $\mathcal{P} \wedge_v \mathcal{E}$, v is a stable Bohl function for all $(w, 0, v) \in \mathcal{P} \wedge_v \mathcal{E}$. Therefore we have $\lim_{t\to\infty}(w(t), v(t)) = 0$ for all $(w, 0, v) \in \mathcal{P} \wedge_v \mathcal{E}$, in other words, (w, v) is detectable from c in $\mathcal{P} \wedge_v \mathcal{E}$. This proves condition 1.
(2) Let $\mathcal{C} = \ker(C\left(\frac{d}{dt}\right))$ be a minimal representation of a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$. From Definition 4.4 and using Theorem 4.3, $\mathcal{N}_{(w,c)}(\mathcal{P})$ is stabilizable. This proves condition 2.
(3) In order to show that condition 3. is necessary for the existence of a regulator we make use of the internal model principle given in Theorem 5.2.

We have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_1\left(\frac{d}{dt}\right) & R_2\left(\frac{d}{dt}\right) & R_3\left(\frac{d}{dt}\right) \\ 0 & C\left(\frac{d}{dt}\right) & 0 \end{bmatrix}\right). \tag{21}$$

The facts that v is free in $\mathcal{P} \wedge_c \mathcal{C}$ and that $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ is stable imply that $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ is Hurwitz. There exists a unimodular matrix U such that

$$U\begin{bmatrix} R_1 & R_2 & R_3 \\ 0 & C & 0 \end{bmatrix} = \begin{bmatrix} \tilde{R}_{11} & 0 & \tilde{R}_{13} \\ \tilde{R}_{21} & \tilde{R}_{22} & \tilde{R}_{23} \end{bmatrix}, \tag{22}$$

where $\tilde{R}_{11}$ and $\tilde{R}_{22}$ are Hurwitz. Therefore we have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} \tilde{R}_{11}\left(\frac{d}{dt}\right) & 0 & \tilde{R}_{13}\left(\frac{d}{dt}\right) \\ \tilde{R}_{21}\left(\frac{d}{dt}\right) & \tilde{R}_{22}\left(\frac{d}{dt}\right) & \tilde{R}_{23}\left(\frac{d}{dt}\right) \end{bmatrix}\right), \tag{23}$$

$$(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} = \ker\left(\begin{bmatrix} \tilde{R}_{11}\left(\frac{d}{dt}\right) & \tilde{R}_{13}\left(\frac{d}{dt}\right) \end{bmatrix}\right), \quad \text{and} \tag{24}$$

$$\mathcal{N}_v((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}) = \ker\left(\tilde{R}_{13}\left(\frac{d}{dt}\right)\right). \tag{25}$$

In order to proceed we need the following lemma:

Lemma 5.10. *Let $A \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{p}}$ be Hurwitz and $B \in \mathbb{R}[\xi]^{\mathtt{q}\times\mathtt{q}}$ be anti-Hurwitz. Then, for any $C \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{q}}$, there exists a solution (X, Y) of the equation $AX + YB = C$.*

Proof. Using the Smith forms of A, B and C, the equation can be reduced to pq scalar B'ezout equations, $ax + by = c$, which all have a solution due to coprimeness of a and b. For details we refer to Lemma 5.3.9 of Fiaz (2010). □

We continue with the proof of Theorem 5.9. From Lemma 5.10, since $\tilde{R}_{22}$ is Hurwitz and V is anti-Hurwitz, there exists a solution $(X, \tilde{Y}_2)$ of the equation

$$\tilde{R}_{22}X + \tilde{R}_{23} = \tilde{Y}_2 V. \tag{26}$$

From Eqs. (20) and (23), we have $\mathcal{P} \wedge_v \mathcal{E} \wedge_c$

$$\mathcal{C} = \ker\left(\begin{bmatrix} \tilde{R}_{11}\left(\frac{d}{dt}\right) & 0 & \tilde{R}_{13}\left(\frac{d}{dt}\right) \\ \tilde{R}_{21}\left(\frac{d}{dt}\right) & \tilde{R}_{22}\left(\frac{d}{dt}\right) & \tilde{R}_{23}\left(\frac{d}{dt}\right) \\ 0 & 0 & V\left(\frac{d}{dt}\right) \end{bmatrix}\right).$$

It is easy to see that $\begin{bmatrix} \tilde{R}_{11}\left(\frac{d}{dt}\right) & 0 & \tilde{R}_{13}\left(\frac{d}{dt}\right) \\ \tilde{R}_{21}\left(\frac{d}{dt}\right) & \tilde{R}_{22}\left(\frac{d}{dt}\right) & \tilde{R}_{23}\left(\frac{d}{dt}\right) \\ 0 & 0 & V\left(\frac{d}{dt}\right) \end{bmatrix}$ has full row rank. Then we have $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{(w,v)} = (\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E} = \ker\left(\begin{bmatrix} \tilde{R}_{11}\left(\frac{d}{dt}\right) & \tilde{R}_{13}\left(\frac{d}{dt}\right) \\ 0 & V\left(\frac{d}{dt}\right) \end{bmatrix}\right)$. From Theorem 5.2, the internal model principle, the fact that $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w = ((\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{(w,v)})_w = ((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)} \wedge_v \mathcal{E})_w$ is stable implies that $\mathcal{E} \subseteq \mathcal{N}_v((\mathcal{P} \wedge_c \mathcal{C})_{(w,v)})$. Hence from Eqs. (20) and (25) there exists a polynomial matrix $\tilde{Y}_1$ such that

$$\tilde{R}_{13} = \tilde{Y}_1 V. \tag{27}$$

Using Eqs. (26) and (27) we have

$$\begin{bmatrix} 0 \\ \tilde{R}_{22} \end{bmatrix} X + \begin{bmatrix} \tilde{R}_{13} \\ \tilde{R}_{23} \end{bmatrix} = \begin{bmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{bmatrix} V. \tag{28}$$

Pre-multiplying both sides by U^{-1} in the above equation, we obtain

$$\begin{bmatrix} R_2 \\ C \end{bmatrix} X + \begin{bmatrix} R_3 \\ 0 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} V \tag{29}$$

where $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} := U^{-1} \begin{bmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{bmatrix}$. Then we have

$$R_2 X + R_3 = Y_1 V, \tag{30}$$

and $CX = Y_2 V$. Therefore, in order to be a regulator, the controller $\mathcal{C}$ must have the internal model of $\mathcal{E}$ in the form of $CX = Y_2 V$.

Since $\mathcal{E} = \ker(V(\frac{d}{dt}))$, from Eq. (30), $\begin{bmatrix} R_2 & R_3 \end{bmatrix} \begin{bmatrix} X\left(\frac{d}{dt}\right) v \\ v \end{bmatrix} = 0$ holds for all $v \in \mathcal{E}$, i.e.,

$$\left(0, X\left(\frac{d}{dt}\right) v, v\right) \in \mathcal{P}.$$

(if) Let $\mathcal{P}$ be given by Eq. (19). There exists a unimodular matrix T such that

$$T \begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \end{bmatrix}, \tag{31}$$

where R_{11} has full row rank. Therefore we have

$$\mathcal{P} = \ker\left(\begin{bmatrix} R_{11}\left(\frac{d}{dt}\right) & R_{12}\left(\frac{d}{dt}\right) & R_{13}\left(\frac{d}{dt}\right) \\ 0 & R_{22}\left(\frac{d}{dt}\right) & R_{23}\left(\frac{d}{dt}\right) \end{bmatrix}\right), \tag{32}$$

$$\mathcal{N}_{(w,c)}(\mathcal{P}) = \ker\left(\begin{bmatrix} R_{11}\left(\frac{d}{dt}\right) & R_{12}\left(\frac{d}{dt}\right) \\ 0 & R_{22}\left(\frac{d}{dt}\right) \end{bmatrix}\right), \tag{33}$$

$$(\mathcal{N}_{(w,c)}(\mathcal{P}))_c = \ker\left(R_{22}\left(\frac{d}{dt}\right)\right), \quad \text{and} \tag{34}$$

$$\mathcal{P} \wedge_v \mathcal{E} = \ker\left(\begin{bmatrix} R_{11}\left(\frac{d}{dt}\right) & R_{12}\left(\frac{d}{dt}\right) & R_{13}\left(\frac{d}{dt}\right) \\ 0 & R_{22}\left(\frac{d}{dt}\right) & R_{23}\left(\frac{d}{dt}\right) \\ 0 & 0 & V\left(\frac{d}{dt}\right) \end{bmatrix}\right). \tag{35}$$

There exists a polynomial matrix $X \in \mathbb{R}[\xi]^{c \times v}$ such that $(0, X\left(\frac{d}{dt}\right) v, v) \in \mathcal{P}$ for all $v \in \mathcal{E}$. Hence $V\left(\frac{d}{dt}\right) v = 0$ implies

$$\begin{bmatrix} R_{12}\left(\frac{d}{dt}\right) \\ R_{22}\left(\frac{d}{dt}\right) \end{bmatrix} X\left(\frac{d}{dt}\right) v + \begin{bmatrix} R_{13}\left(\frac{d}{dt}\right) \\ R_{23}\left(\frac{d}{dt}\right) \end{bmatrix} v = 0. \tag{36}$$

Therefore there exists a polynomial matrix $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ such that

$$\begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix} X + \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} V. \tag{37}$$

This implies

$$R_{22} X + R_{23} = Y_2 V. \tag{38}$$

From Eq. (33), the fact that $\mathcal{N}_{(w,c)}(\mathcal{P})$ is stabilizable implies that $\begin{bmatrix} R_{11}(\lambda) & R_{12}(\lambda) \\ 0 & R_{22}(\lambda) \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$, which in turn implies that $R_{22}(\lambda)$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$. From Eq. (34), we conclude that $(\mathcal{N}_{(w,c)}(\mathcal{P}))_c$ is stabilizable. From Proposition 3.3 there exists a $\mathcal{C} \in \mathfrak{L}^c$ such that $(\mathcal{N}_{(w,c)}(\mathcal{P}))_c \cap \mathcal{C}$ is stable and regular. Factor R_{22} as $R_{22} = DK$ where D is Hurwitz and $K(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Let S be such that $\begin{bmatrix} K \\ S \end{bmatrix}$ is unimodular. Then for an arbitrary polynomial matrix F and an arbitrary Hurwitz polynomial matrix H of suitable dimensions, it is easy to verify that

$$C = FR_{22} + HS \tag{39}$$

serves as a stabilizing controller for $(\mathcal{N}_{(w,c)}(\mathcal{P}))_c$. Note that $\begin{bmatrix} R_{22} \\ C \end{bmatrix}$ is Hurwitz for all C given by the Eq. (39).

From Eq. (35), (w, v) is detectable from c in $\mathcal{P} \wedge_v \mathcal{E}$ implies that $\begin{bmatrix} R_{11}(\lambda) & R_{13}(\lambda) \\ 0 & R_{23}(\lambda) \\ 0 & V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. This implies that R_{11} is square nonsingular and Hurwitz and $\begin{bmatrix} R_{23}(\lambda) \\ V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. As $V(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}^-$ (use the fact that V is anti-Hurwitz) we conclude that $\begin{bmatrix} R_{23}(\lambda) \\ V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$. Hence there exists a solution (F, M) of the equation

$$FR_{23} + MV = HSX. \tag{40}$$

We now prove that any controller given by $\mathcal{C} = \ker(C\left(\frac{d}{dt}\right))$ where $C = FR_{22} + HS$ with F satisfying Eq. (40) serves as a regulator. The following identities hold true.

$$\begin{aligned} CX &= FR_{22}X + HSX \\ &= FR_{22}X + FR_{23} + MV \text{(from Eq. (40))} \\ &= F(R_{22}X + R_{23}) + MV \\ &= FY_2V + MV \text{(from Eq. (38))} \\ &= (FY_2 + M)V. \end{aligned}$$

Then, we define $W := FY_2 + M$ to rewrite the above equality as

$$CX = WV. \tag{41}$$

We also have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker\left(\begin{bmatrix} R_{11}\left(\frac{d}{dt}\right) & R_{12}\left(\frac{d}{dt}\right) & R_{13}\left(\frac{d}{dt}\right) \\ 0 & R_{22}\left(\frac{d}{dt}\right) & R_{23}\left(\frac{d}{dt}\right) \\ 0 & C\left(\frac{d}{dt}\right) & 0 \end{bmatrix}\right), \tag{42}$$

$$\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C}) = \ker\left(\begin{bmatrix} R_{11}\left(\frac{d}{dt}\right) & R_{12}\left(\frac{d}{dt}\right) \\ 0 & R_{22}\left(\frac{d}{dt}\right) \\ 0 & C\left(\frac{d}{dt}\right) \end{bmatrix}\right). \tag{43}$$

As C is chosen such that $\begin{bmatrix} R_{22} \\ C \end{bmatrix}$ is Hurwitz, $\begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & C \end{bmatrix}$ is square, nonsingular and Hurwitz. Hence, the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular from Eq. (42), and $\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C})$ is stable from Eq. (43). It also follows that v is free in $\mathcal{P} \wedge_c \mathcal{C}$. We have

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \left\{ (w, c, v) \left| \begin{array}{l} R_{11}\left(\frac{d}{dt}\right) w + R_{12}\left(\frac{d}{dt}\right) c + R_{13}\left(\frac{d}{dt}\right) v = 0, \\ R_{22}\left(\frac{d}{dt}\right) c + R_{23}\left(\frac{d}{dt}\right) v = 0, \\ C\left(\frac{d}{dt}\right) c = 0, V\left(\frac{d}{dt}\right) v = 0 \end{array} \right. \right\}.$$

Substituting Eq. (37) into the above equation yields

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \left\{ (w,c,v) \left| \begin{array}{l} R_{11}\left(\frac{d}{dt}\right) w + R_{12}\left(\frac{d}{dt}\right)\left(c - X\left(\frac{d}{dt}\right)v\right) \\ +Y_1 V\left(\frac{d}{dt}\right) v = 0, \\ R_{22}\left(\frac{d}{dt}\right)\left(c - X\left(\frac{d}{dt}\right)v\right) + Y_2 V\left(\frac{d}{dt}\right) v = 0, \\ C\left(\frac{d}{dt}\right)\left(c - X\left(\frac{d}{dt}\right)v\right) + CX\left(\frac{d}{dt}\right) v = 0, \\ V\left(\frac{d}{dt}\right) v = 0 \end{array} \right. \right\}.$$

It further follows from Eq. (41) that

$$\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \left\{ (w,c,v) \left| \begin{array}{l} R_{11}\left(\frac{d}{dt}\right) w + R_{12}\left(\frac{d}{dt}\right)\left(c - X\left(\frac{d}{dt}\right)v\right) \\ +Y_1 V\left(\frac{d}{dt}\right) v = 0, \\ R_{22}\left(\frac{d}{dt}\right)\left(c - X\left(\frac{d}{dt}\right)v\right) + Y_2 V\left(\frac{d}{dt}\right) v = 0, \\ C\left(\frac{d}{dt}\right)\left(c - X\left(\frac{d}{dt}\right)v\right) + WV\left(\frac{d}{dt}\right) v = 0, \\ V\left(\frac{d}{dt}\right) v = 0 \end{array} \right. \right\}$$

$$= \left\{ (w,c,v) \left| \begin{array}{l} R_{11}\left(\frac{d}{dt}\right) w + R_{12}\left(\frac{d}{dt}\right)\left(c - X\left(\frac{d}{dt}\right)v\right) = 0, \\ R_{22}\left(\frac{d}{dt}\right)\left(c - X\left(\frac{d}{dt}\right)v\right) = 0, \\ C\left(\frac{d}{dt}\right)\left(c - X\left(\frac{d}{dt}\right)v\right) = 0, \\ V\left(\frac{d}{dt}\right) v = 0 \end{array} \right. \right\}.$$

From the above, we see that, for all $(w, c, v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$, $(w, c - X\left(\frac{d}{dt}\right) v)$ belongs to $\ker\left(\begin{bmatrix} R_{11}\left(\frac{d}{dt}\right) & R_{12}\left(\frac{d}{dt}\right) \\ 0 & R_{22}\left(\frac{d}{dt}\right) \\ 0 & C\left(\frac{d}{dt}\right) \end{bmatrix}\right)$.

Since $\begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \\ 0 & C \end{bmatrix}$ is Hurwitz, $\lim_{t\to\infty}(w(t), c(t) - X\left(\frac{d}{dt}\right) v(t)) = 0$ holds for all $(w, c, v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$. This clearly implies that $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w$ is stable. This completes the proof of Theorem 5.9. □

Remark 5.11. Conditions (1) and (2) of Theorem 5.9 are self explanatory. From the if part of the proof it is clear that the condition 3. along with conditions (1) and (2) guarantees the existence of a controller $\mathcal{C} = \ker(C\left(\frac{d}{dt}\right))$ with the structure $CX = WV$, which guarantees the internal model principle for $(\mathcal{P} \wedge_c \mathcal{C})_{(w,v)}$ (see Remark 5.3).

In this remark we also give an alternative interpretation of condition 3. In the following, two behaviors $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathrm{w}}$ will be called isomorphic if there exists a unimodular matrix U such that $\mathfrak{B}_1 = U\left(\frac{d}{dt}\right)\mathfrak{B}_2$. If $R_1\left(\frac{d}{dt}\right) w = 0$ and $R_2\left(\frac{d}{dt}\right) w = 0$ are kernel representations (not necessarily minimal) of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, respectively, where $R_1, R_2 \in \mathbb{R}[\xi]^{\mathrm{q}\times\mathrm{w}}$, then obviously $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are isomorphic if and only if there exist unimodular matrices T and U such that $TR_1U = R_2$, equivalently, R_1 and R_2 have the same Smith form.

Now, let $\mathcal{P}$ and $\mathcal{E}$ be given by the minimal kernel representations $R_1\left(\frac{d}{dt}\right) w + R_2\left(\frac{d}{dt}\right) c + R_3\left(\frac{d}{dt}\right) v = 0$ and $V\left(\frac{d}{dt}\right) v = 0$, respectively. It is easily seen that condition (3) of Theorem 5.9 is equivalent to solvability of the polynomial Sylvester equation

$$R_2 X + R_3 = YV \tag{44}$$

in the unknown (X, Y). From Hautus (1983) (also see Theorem 9.7 in Trentelman et al., 2001) Eq. (44) is solvable for (X, Y) if and only if the matrices

$$\begin{bmatrix} R_2 & 0 \\ 0 & V \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} R_2 & R_3 \\ 0 & V \end{bmatrix}$$

have the same Smith form, i.e., there exist unimodular matrices T and U such that

$$T\begin{bmatrix} R_2 & 0 \\ 0 & V \end{bmatrix} U = \begin{bmatrix} R_2 & R_3 \\ 0 & V \end{bmatrix}. \tag{45}$$

We will now give a behavioral interpretation of this condition. Clearly,

$$\mathcal{N}_{(c,v)}(\mathcal{P}) \wedge_v \mathcal{E} = \ker\left(\begin{bmatrix} R_2\left(\frac{d}{dt}\right) & R_3\left(\frac{d}{dt}\right) \\ 0 & V\left(\frac{d}{dt}\right) \end{bmatrix}\right), \quad \text{and} \tag{46}$$

$$\mathcal{N}_c(\mathcal{P}) \times \mathcal{E} = \ker\left(\begin{bmatrix} R_2\left(\frac{d}{dt}\right) & 0 \\ 0 & V\left(\frac{d}{dt}\right) \end{bmatrix}\right). \tag{47}$$

Thus, condition 3. requires that the behaviors $\mathcal{N}_c(\mathcal{P}) \times \mathcal{E}$ and $\mathcal{N}_{(c,v)}(\mathcal{P}) \wedge_v \mathcal{E}$ are isomorphic. The behaviors $\mathcal{N}_{(c,v)}(\mathcal{P}) \wedge_v \mathcal{E}$ and $\mathcal{N}_c(\mathcal{P}) \times \mathcal{E}$ are shown schematically in Fig. 7(a) and (b), respectively. Note that $\mathcal{N}_{(c,v)}(\mathcal{P}) \wedge_v \mathcal{E}$ is the (c, v)-behavior in the interconnection of the plant and the exosystem with $w = 0$, while $\mathcal{N}_c(\mathcal{P}) \times \mathcal{E}$ is the (c, v)-behavior in the disconnected system of the plant with $(w, v) = (0, 0)$ and the exosystem. Thus the condition can be interpreted as requiring that the behavior obtained after disconnecting the plant and the exosystem (with $w = 0$) are isomorphic.

We will now outline an algorithmic procedure that, starting with polynomial kernel representations of $\mathcal{P} \in \mathfrak{L}^{\mathrm{w+c+v}}$ and $\mathcal{E} \in \mathfrak{L}^{\mathrm{v}}_{\mathrm{aut}}$, checks whether a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ exists. If there exists a regulator, the algorithm also gives a procedure to construct one.

Algorithm 1. Let $R_1\left(\frac{d}{dt}\right) w + R_2\left(\frac{d}{dt}\right) c + R_3\left(\frac{d}{dt}\right) v = 0$ and $V\left(\frac{d}{dt}\right) v = 0$ be minimal kernel representations of $\mathcal{P}$ and $\mathcal{E}$, respectively, where $\begin{bmatrix} R_1 & R_2 \end{bmatrix}$ has full row rank and V is square and nonsingular. Then,

(1) If $\begin{bmatrix} R_1(\lambda) & R_2(\lambda) \end{bmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$ continue further, else declare there exists no regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.
(2) If $R_1(\lambda)$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$ continue further, else declare there exists no regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.
(3) If $\begin{bmatrix} R_3(\lambda) \\ V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$ continue further, else factorize

$$\begin{bmatrix} R_3 \\ V \end{bmatrix} = \begin{bmatrix} R_3' \\ V' \end{bmatrix} D,$$

where D is square and nonsingular and $\begin{bmatrix} R_3'(\lambda) \\ V'(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$.
Assign $R_3 = R_3'$ and $V = V'$.
(4) If $\begin{bmatrix} R_1(\lambda) & R_3(\lambda) \\ 0 & V(\lambda) \end{bmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$ continue further, else declare there exists no regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.

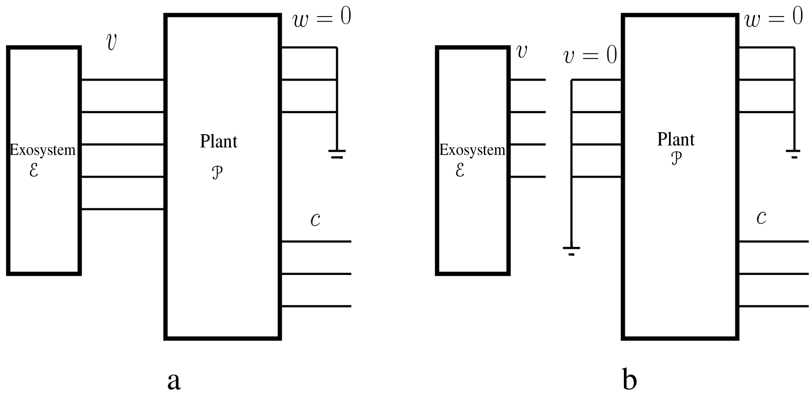


Fig. 7. (a) $\mathcal{N}_{(c,v)}(\mathcal{P}) \wedge_v \mathcal{E}$. (b) $\mathcal{N}_c(\mathcal{P}) \times \mathcal{E}$.

(5) If V is anti-Hurwitz continue further, else factorize $V = U_1 \sum_- \sum_+ U_2$ where U_1, U_2 are unimodular matrices and $\sum_-$, $\sum_+$ are diagonal polynomial matrices such that $\sum_-$ is Hurwitz and $\sum_+$ is anti-Hurwitz. Assign $V = \sum_+ U_2$.

(6) Solve

$$R_2 X + R_3 = YV \tag{48}$$

for (X, Y). If there exists no solution, declare there exists no regulator for $\mathcal{P}$ with respect to $\mathcal{E}$, else continue further.

(7) Choose a unimodular matrix T such that

$$T \begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \end{bmatrix}, \tag{49}$$

where R_{11} has full row rank. Factor R_{22} as $R_{22} = D_1 K$ where D_1 is Hurwitz and $K(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Choose S such that $\begin{bmatrix} K \\ S \end{bmatrix}$ is unimodular.

(8) Solve

$$\begin{bmatrix} F & M \end{bmatrix} \begin{bmatrix} R_{23} \\ V \end{bmatrix} = HSX \tag{50}$$

for (F, M), where H is an arbitrary Hurwitz polynomial matrix.

(9) Define $C := FR_{22} + HS$. Then the controller $\mathcal{C}$ defined by $C\left(\frac{d}{dt}\right) c = 0$ is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.

In order to illustrate the theory developed so far in this paper we now present some worked-out examples.

Example 5.12. Let the plant $\mathcal{P}$, with to-be-regulated variable w, interconnection variable (c_1, c_2) and disturbance variable v, be given by

$$\begin{bmatrix} 1 & \frac{d}{dt}+3 & 1 & \frac{d}{dt}+1 \\ \frac{d}{dt}+2 & 0 & 0 & \frac{d}{dt}+4 \end{bmatrix} \begin{bmatrix} w \\ c_1 \\ c_2 \\ v \end{bmatrix} = 0.$$

Let the exosystem $\mathcal{E}$ with system variable v be given by

$$\frac{d}{dt} v - v = 0. \tag{51}$$

Then $\mathcal{N}_{(w,c_1,c_2)}(\mathcal{P})$ and $\mathcal{P} \wedge_v \mathcal{E}$ are given by

$$\begin{bmatrix} 1 & \frac{d}{dt}+3 & 1 \\ \frac{d}{dt}+2 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ c_1 \\ c_2 \end{bmatrix} = 0$$

and

$$\begin{bmatrix} 1 & \frac{d}{dt}+3 & 1 & \frac{d}{dt}+1 \\ \frac{d}{dt}+2 & 0 & 0 & \frac{d}{dt}+4 \\ 0 & 0 & 0 & \frac{d}{dt}-1 \end{bmatrix} \begin{bmatrix} w \\ c_1 \\ c_2 \\ v \end{bmatrix} = 0$$

respectively.

(1) It is easy to see that w is detectable from (c_1, c_2, v) in $\mathcal{P}$ and $\mathcal{N}_{(w,c_1,c_2)}(\mathcal{P})$ is stabilizable. Therefore from Theorem 4.3 there exists a regular, admissible, stabilizing controller for $\mathcal{P}$. It is easy to verify that $\mathcal{C} = \{(c_1, c_2) \mid c_1 = 0\}$ is a regular, admissible, stabilizing controller for $\mathcal{P}$.

(2) It is also easy to see that $\mathcal{E}$ is an anti-stable system, v is observable from (w, c_1, c_2) in $\mathcal{P} \wedge_v \mathcal{E}$ and (w, v) is detectable from (c_1, c_2) in $\mathcal{P} \wedge_v \mathcal{E}$. There exists a polynomial matrix $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}[\xi]^{2\times 1}$ such that $(0, X_1\left(\frac{d}{dt}\right) v, X_2\left(\frac{d}{dt}\right) v, v) \in \mathcal{P}$ for all $v \in \mathcal{E}$ if and only if there exist polynomial matrices $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}[\xi]^{2\times 1}$ and $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \in \mathbb{R}[\xi]^{2\times 1}$ satisfying the equation

$$\begin{bmatrix} \xi+3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} \xi+1 \\ \xi+4 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} (\xi - 1). \tag{52}$$

As $\xi + 4 = Y_2(\xi - 1)$ is not solvable for $Y_2 \in \mathbb{R}[\xi]$, Eq. (52) is also not solvable. Therefore from Theorem 5.9 there does not exists a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.

Example 5.13. Let the plant $\mathcal{P}$, with to-be-regulated variable w, interconnection variable (c_1, c_2) and disturbance variable v, and the exosystem $\mathcal{E}$ with system variable v be given by

$$\begin{bmatrix} R_{11}\left(\frac{d}{dt}\right) & R_{12}\left(\frac{d}{dt}\right) & R_{13}\left(\frac{d}{dt}\right) \\ 0 & R_{22}\left(\frac{d}{dt}\right) & R_{23}\left(\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} w \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ v \end{bmatrix} = 0$$

$$V\left(\frac{d}{dt}\right) v = 0,$$

where $R_{11} = \xi + 2$, $R_{12} = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $R_{13} = \xi + 1$, $R_{22} = \begin{bmatrix} \xi - 2 & -1 \end{bmatrix}$, $R_{23} = -\xi$ and $V = \xi - 1$.

(1) It is easy to see that w is detectable from (c_1, c_2, v) in $\mathcal{P}$ and $\mathcal{N}_{(w,c_1,c_2)}(\mathcal{P})$ is stabilizable. Therefore from Theorem 4.3 there exists a regular, admissible, stabilizing controller for $\mathcal{P}$. It is easy to verify that $\mathcal{C} = \{(c_1, c_2) \mid c_1 = 0\}$ is a regular, admissible, stabilizing controller for $\mathcal{P}$.

(2) It is also easy to see that $\mathcal{E}$ is an anti-stable system, v is observable from (w, c_1, c_2) in $\mathcal{P} \wedge_v \mathcal{E}$ and (w, v) is detectable from (c_1, c_2) in $\mathcal{P} \wedge_v \mathcal{E}$. There exists a polynomial matrix $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}[\xi]^{2\times 1}$ such that $(0, X_1\left(\frac{d}{dt}\right)v, X_2\left(\frac{d}{dt}\right)v, v) \in \mathcal{P}$ for all $v \in \mathcal{E}$ if and only if there exist polynomial matrices $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}[\xi]^{2\times 1}$ and $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \mathbb{R}[\xi]^{2\times 1}$ satisfying the equation

$$\begin{bmatrix} 0 & 1 \\ \xi - 2 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} \xi + 1 \\ -\xi \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} (\xi - 1). \tag{53}$$

It is easy to see that $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a solution to Eq. (53). Therefore, from Theorem 5.9 there exists a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$. We note here that the controller $\mathcal{C} = \{(c_1, c_2) \mid c_1 = 0\}$ is a regular, admissible, stabilizing controller for $\mathcal{P}$ but not a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.

Now we use Algorithm-1 to construct a regular, admissible, stabilizing controller of $\mathcal{P}$ which also acts as a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$. As the conditions in steps 1–6 of Algorithm-1 are already satisfied, we here start from step 7. of Algorithm-1.

7. As $R_{22}(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$, we have $K = R_{22}$ and $D = 1$. Then S defined by $S := \begin{bmatrix} 1 & 0 \end{bmatrix}$ satisfies the condition that $\begin{bmatrix} K \\ S \end{bmatrix}$ is unimodular.
8. For the choice $H = 1$, we have $HSX = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1$. Then the solution to Eq. (50) is given by $\begin{bmatrix} F & M \end{bmatrix} = \begin{bmatrix} -1 & -1 \end{bmatrix}$.
9. Then $C = FR_{22} + HS = -1\begin{bmatrix} \xi - 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -\xi + 3 & 1 \end{bmatrix}$. The controller defined by

$$\mathcal{C} = \left\{ (c_1, c_2) \mid -\frac{d}{dt}c_1 + 3c_1 - c_2 = 0 \right\}$$

is a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.

Remark 5.14. In the problem formulation of this paper, the plant contains three kinds of variables, namely the variable w to be regulated, the control variable c, and the exogenous variable v. A possible extension is to include in the plant an additional variable, called w' that (like c) only needs to be taken to rest if the exogenous variable v is equal to zero. In this new setup, our plant $\mathcal{P}$ has variables (w', w, c, v), where v is free. The aim it to find a "modified regulator" for $\mathcal{P}$, which is defined to be a regular controller $\mathcal{C} \in \mathfrak{L}^c$ such that

(1) v is free in $\mathcal{P} \wedge_c \mathcal{C}$,
(2) $(w', w, c, 0) \in \mathcal{P} \wedge_c \mathcal{C} \implies (w'(t), w(t), c(t)) \to 0$ as $t \to \infty$,
(3) $(w', w, c, v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} \implies w(t) \to 0$ as $t \to \infty$.

It can be shown that this new problem can be reduced to the problem studied in this paper. This can be done by eliminating the new variable w' from $\mathcal{P}$, thus obtaining the system $(\mathcal{P})_{(w,c,v)} \in \mathfrak{L}^{w+c+v}$. It can then be shown that $\mathcal{C} \in \mathfrak{L}^c$ is a "modified regulator" for the extended plant $\mathcal{P}$ if and only if $\mathcal{C}$ is a regulator (in the sense of this paper) for the projected plant $(\mathcal{P})_{(w,c,v)}$. For details, we refer to Fiaz (2010).

The above allows us to apply the results of this paper to the classical regulator problem in the input-state–output setting (see Francis, 1977; Francis & Wonham, 1975). In that case, apart from the to be regulated output w, the control variable (u, y) and the exogenous variable v, the plant contains the state variable x which has to driven to zero if the exogenous signal v is equal to zero. The 'classical' results in this context can thus be reobtained by applying the results from this paper. Again, for details we refer to Fiaz (2010).

6. Conclusions

In this paper we have formulated and resolved the problem of asymptotic tracking and regulation in a completely representation-free manner. We have used the theory of behavioral control for this purpose. In the behavioral context, controllers act on the plant using general interconnection, without *a priori* input–output partitions. Given a plant and an exosystem, we have established necessary and sufficient conditions for the existence of a regulator only in terms of the plant and exosystem dynamics.

References

Belur, M. N., & Trentelman, H. L. (2002). Stabilization, pole placement and regular implementability. *IEEE Transactions on Automatic Control*, *47*(5), 735–744.

Belur, M.N. Control in a behavioral context, doctoral dissertation, University of Groningen, The Netherlands, 2003.

Bisiacco, M., & Valcher, M. E. (2001a). Behavior decompositions and two-sided diophantine equations. *Automatica*, *37*(9), 1387–1395.

Bisiacco, M., & Valcher, M. E. (2001b). Autonomous behaviors decomposition and modal analysis. *International Journal of Control*, *74*(17), 1690–1705.

Davison, E. J. (1975). The output control of linear time-invariant multi-variable systems with unmeasured arbitrary disturbances. *IEEE Transactions on Automatic Control*, *20*(12), 824.

Davison, E. J., & Goldenberg, A. (1975). The robust control of a general servomechanism problem: the servo compensator. *Automatica*, *11*, 461–471.

Fiaz, S., Takaba, K., & Trentelman, H.L. Tracking and regulation in the behavioral framework, In Proceedings of mathematical theory of networks and system, Budapest, Hungary, 2010.

Fiaz, S. Regulation and robust stabilization: a behavioral approach, Doctoral dissertation, University of Groningen, The Netherlands, 2010.

Francis, B. A. (1977). The linear multivariable regulator problem. *SIAM Journal on Control and Optimization*, *15*(3), 486–505.

Francis, B. A., & Wonham, W. M. (1975). The internal model principle for linear multivariable regulators. *Applied Mathematics and Optimization*, *2*(2), 170–194.

Hautus, M.L.J. Linear matrix equations with applications to the regulator problem, Outils et Modeles Mathématique pour l'Automatique, I.D. Landau, ed., C.N.R.S., Paris, pp. 399–412, 1983.

Isidori, A., & Byrnes, C. I. (1990). Output regulation of nonlinear systems. *IEEE Transactions on Automatic Control*, *35*(2), 131–140.

Polderman, J. W., & Willems, J. C. (1997). *Introduction to mathematical systems theory: A behavioral approach*. Berlin: Springer-Verlag.

Saberi, A., Stoorvogel, A. A., & Sannuti, P. (2000). Control of linear systems with regulation and input constraints. In *Communication and control engineering series*. Springer Verlag.

Takaba, K. A note on the regulation problem in the behavioral Framework, In Proceedings of ICROS-SICE International Joint Conference 2009, Fukuoka, Japan, August 18–21, 2009.

Trentelman, H. L., Stoorvogel, A. A., & Hautus, M. L. J. (2001). *Control theory for linear systems*. London: Springer.

Willems, J. C. (1997). On interconnection, control, and feedback. *IEEE Transactions on Automatic Control*, *42*, 326–339.

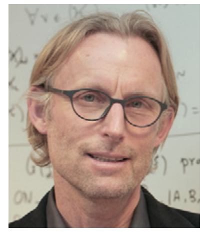
Willems, J. C., & Trentelman, H. L. (2002). Synthesis of dissipative systems using quadratic differential forms-part I. *IEEE Transactions on Automatic Control*, *47*(1), 53–69.

Shaik Fiaz was born in Macherla, India. His pre-university years were spent in Jogipet, India, followed by the Diploma in Electrical Engineering from the Government Polytechnic (GPT), Hyderabad, India in 2001. He received the B.Tech. degree in Electrical Engineering from the Jawaharlal Nehru Technological University (JNTU), Hyderabad, India, in 2004, the M.Tech. degree in Electrical Engineering from the Indian Institute of Technology (IIT), Bombay campus, Mumbai, India in 2006, the Ph.D. degree in Systems and Control from the University of Groningen, The Netherlands, in 2010.

Currently he is working as a post-doctoral fellow in the research groups of Discrete Technology and Production Automation, and Systems and Control at the University of Groningen, The Netherlands. His research interests include control in the behavioral framework, power systems, dynamic networks, networked control systems, distributed control of multi-agent systems, fault diagnosis and discrete event systems.

Dr. Shaik Fiaz received the Dutch Institute of Systems and Control (DISC) Best Ph.D. Thesis award for the best PhD thesis defended in 2010 in The Netherlands in the area of systems and control.

K. Takaba received his B.Eng., M.Eng., and Dr.Eng. degrees from Kyoto University, Japan, in 1989, 1991 and 1996, respectively. In 1991, he joined the faculty of Kyoto University, where he is currently an Associate Professor of Department of Applied Mathematics and Physics. His research interests include robust/optimal control, behavioral approach to systems and control, constrained control systems, and networked control systems. He is a member of IFAC and IEEE.



H.L. Trentelman is a full professor in Systems and Control at the Johann Bernoulli Institute for Mathematics and Computer Science of the University of Groningen in The Netherlands. From 1991 to 2008, he served as an associate professor and later as an adjoint professor at the same institute. From 1985 to 1991, he was an assistant professor, and later, an associate professor at the Mathematics Department of the University of Technology at Eindhoven, The Netherlands. He obtained his Ph.D. degree in Mathematics at the University of Groningen in 1985. His research interests are the behavioral approach to systems and control, robust control, model reduction, multi-dimensional linear systems, hybrid systems, analysis and control of networked systems, and the geometric theory of linear systems. He is a co-author of the textbook "Control Theory for Linear Systems" (Springer, 2001). Dr. Trentelman is an associate editor of the IEEE Transactions on Automatic Control and of Systems and Control Letters, and is past associate editor of the SIAM Journal on Control and Optimization.